Theory of slow, steady state crack growth in polymer glasses

Edward J. Kramer
Department of Materials Science and Engineering and the Materials Science Center, Cornell University, Ithaca, NY 14853, USA

and Edward W. Hart
Departments of Theoretical and Applied Mechanics and Materials Science and Engineering and the Materials Science Center, Cornell University, Ithaca, NY 14853, USA
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Cracks in polymer glasses can grow slowly preceded by a craze, a narrow zone of plastic cavitation. The craze widens by drawing more polymer from its surfaces into its fibrils but the fibrils themselves fail by local creep. When the crack tip moves at velocity v the loading at the crack tip can be described by a *local* stress intensity factor K which is the sum of the 'apparent' stress intensity factor K_A and a plastic contribution K_p (usually negative). K_p is found to be $-KP(K)/v$ where $P(K)$ is an integral over the craze boundary displacement law. Fibril failure by local creep leads to a power law, $v \propto K^m$. From these relations K and v can be determined as a function of K_A. The plot of K vs. K_A is multiple-valued with a stable branch (at high K) and an unstable branch (at low K) separated by a minimum value of K_A which represents a threshold for stable, steady state crack growth. There is also a v threshold, below which cracks will not grow steadily. These predictions, the form of the v–K_A curve and implications for slip–stick crack growth are compared with recent experiments.

(Keywords: steady state crack growth; craze breakdown; polymer glasses)

INTRODUCTION

The slow propagation of cracks in polymer glasses has long been of interest, not just because of its technological importance, but because the crack in such polymers is preceded by a relatively simple zone of plastic deformation, a craze. The craze is a narrow zone of fibrillation that forms almost a linear extension of the crack[1,2]. The craze tapers from its maximum width at the crack tip to typically 10 nm wide at its tip, which advances by a finger-like growth produced by the Taylor meniscus instability[3,4]. A continuous void network is thus created, with small fibrils (typically *ca.* 10 nm diameter) being created at the polymer webs between the void fingers. The craze widens by drawing more polymer from the craze–bulk polymer interfaces into the fibrils, a process termed surface drawing[5–7]. At nearly constant craze surface stress, the drawing process proceeds at an essentially constant extension ratio λ, which is determined by the entanglement density of the polymer, low entanglement densities producing high extension ratios and *vice versa*[8–11].

Crack propagation in polymer glasses under conditions where a single craze exists at the crack tip has been extensively investigated, both experimentally and theoretically. The usual approach is to assume the Dugdale model is applicable, such that the craze may be modelled as a zone that supports a constant tensile stress S_c along its length. The model allows one to predict craze opening displacement profiles ($w(x)$ as well as craze lengths for various conditions of loading (i.e. various stress intensity factors K_A†) for crazes and cracks which are stationary. These predictions may be compared with experimental measurements of these quantities using microscopic optical interference techniques and reasonable agreement is usually obtained[12–16].

For crazes growing ahead of static cracks a similar model is used but S_c and the effective Youngs modulus of the polymer E^* are allowed to be time dependent[17,18]. When the crack is moving, however, a subsidiary assumption is needed. The Dugdale model does not provide a criterion for crack advance. It is usual to assume a critical crack opening displacement or *COD*, a maximum displacement w_c of the two craze interfaces at the tip of the crack. Since it is easily shown from the Dugdale model that the fracture toughness, $G_{I_c} = 2S_c w_c$, this assumption is attractive in that it leads to a G_{I_c} that is almost independent of crack speed v (depending only on $S_c(v)$). Early measurements of w_c also seemed to confirm that it was essentially constant over a fairly wide range of temperature and v[12,15,16].

More recently, however, careful measurements by Doell, Koenczoel and Schinker reveal that by loading cracks in PMMA to K_A's just below a threshold value, *ca.* 0.66 MPam$^{1/2}$, crazes may be grown from non-propagating cracks which have w_c up to 3 times larger than the COD for crazes in front of propagating cra-

† In this paper we shall use the notation K_A for the 'apparent' stress intensity factor, i.e. that conventionally defined by assuming material elastic behaviour. The symbol K with no subscript will be used here for the local actual stress intensity factor as described below.

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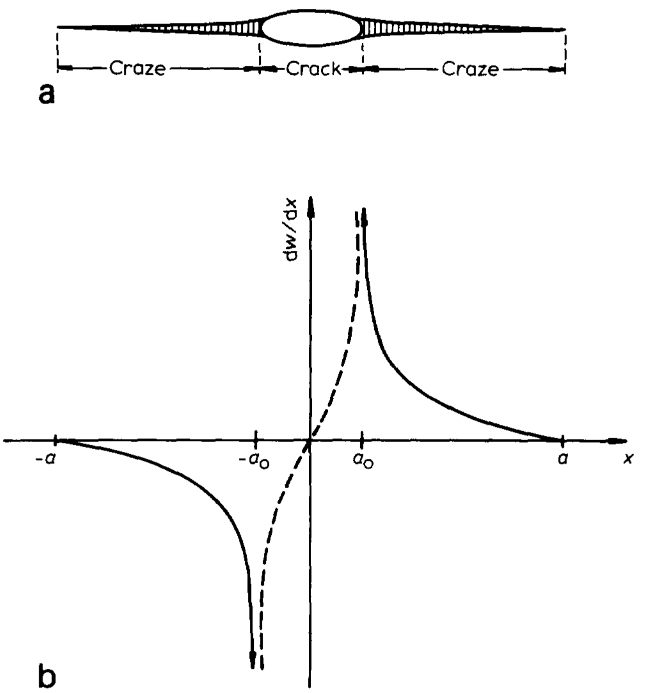


Figure 1 (a) Schematic of crazes growing from crack tips in a polymer glass. (b) Craze surface displacement derivative profile computed from the Dugdale model

cks[20,21]. It is difficult to understand how a true critical opening displacement if it really exists, can be exceeded. It is also hard to understand how a true critical craze opening displacement can exist for crazes which widen by surface drawing. For high molecular weight polymers there should be no natural upper limit for width of a craze produced at a crack tip by surface drawing (drawing at constant λ should not cause the craze to become weaker with increasing width). Any apparent constancy of w_c must be due to other factors.

On purely theoretical grounds, moreover, strong objections can be raised to using the Dugdale model to represent cracks moving at constant velocity. Consider the double-ended crack and craze shown in *Figure 1a*. Let x be a coordinate along the crack craze axis, let a_0 be the crack length and let a be the length of the crack plus craze. From the Dudgale model the displacement derivative $\mathrm{d}w/\mathrm{d}x$ is given by[22]

$$\frac{\mathrm{d}w}{\mathrm{d}x}=\frac{S_c}{2\pi E^*}\ln\frac{\sin^2(\beta_c-\beta)}{\sin^2(\beta_c+\beta)} \qquad (1)$$

where $\cos\beta=x/a$, $\cos\beta_c=a_0/a$ and E^* is an effective Young's modulus ($E^*=E$ for plane stress and $E/(1-\nu^2)$ for plane strain where ν is Poisson's ratio). As can be seen from equation (1) and *Figure 1b*, $\mathrm{d}w/\mathrm{d}x$ is logarithmically singular at the crack tip. This fact causes no special problems while the crack is stationary. When the crack moves at constant velocity v, however, the displacement rate $\dot{w}(x)$ at the craze surfaces is given by

$$\dot{w}(x)=-v\frac{\mathrm{d}w}{\mathrm{d}x} \qquad (2)$$

Thus the displacement rate at the crack tip in the Dugdale model becomes infinite at the crack tip. Since $\dot{w}(x)$, which is proportional to a fibril drawing rate, must be coupled to the local tensile stress across the craze, the strong dependence of $\dot{w}$ on position along the craze makes the Dugdale assumption of a constant S_c untenable for a moving crack.*

There is an additional problem with the Dugdale model that should be considered. Because the Dugdale model describes only plastic deformation the stress at the crack tip is non-singular and no stress intensity factor is defined at the crack tip. Of course there cannot be a stress singularity physically, and so the lack of a singularity is not a flaw of the model. However, there is no parameter in the theory that plays the role of stress intensity factor. It will be recalled that the detailed Barenblatt model[35] for the cracking process does provide an expression for a stress intensity factor K even when the stress is non-singular, and a crack propagation force $G=K^2/E$ is defined. Such a force G is necessary if the cracking is to represent discontinuous separation of material and the production of new crack surface.

An important characteristic of the Barenblatt model is that it shows that the stress state at the crack tip can be treated *as though it had the singularity of linear elasticity* and that the resultant K will be substantially the same as for the model with no singularity.

In the present paper a theory for crack propagation in polymer glasses is presented that is free of the defects of the Dugdale model. It is based on an earlier theory of Hart[23,36] for Mode III crack propagation in metals. In that theory a clear distinction is made between the material deformation process and the cracking process. In the present treatment the deformation process is the highly localized craze growth process which can be modelled as a nearly one-dimensional zone of deformation ahead of the crack. The cracking process is the process of local creep instability and rupture of the craze fibrils in a zone‡ close to the crack tip within the already drawn craze. This time dependent cracking process replaces the static crack propagation criteria of standard fracture mechanics and allows the definition of a crack propagation force in the sense of the Barenblatt model. Because of the one-dimensional geometry of craze and crack, the Mode III treatment of Hart is easily transformed to a Mode I configuration.

OUTLINE OF THE THEORY

Consider the semi-infinite crack moving slowly at velocity v shown in *Figure 2a*. Let x be a coordinate along the extension of the crack with its zero at the moving crack tip. In the absence of inelastic deformation ahead of the crack we may compute the stress σ^{A} ahead of the crack from the applied stress intensity factor K_{A}, i.e.,

$$\sigma^{\mathrm{A}}(x)=K_{\mathrm{A}}/\sqrt{2\pi x} \qquad (3)$$

These stresses, however, will be relaxed by plastic deformation in the zone ahead of the crack, in this case a craze. It is useful to view the final stress state as a superposition of the stresses of the purely elastic body, equation (3) and

* It might be imagined that simply increasing the stresses along the craze toward the crack tip would eliminate the singularity in $\mathrm{d}w/\mathrm{d}x$. It will not. It can be shown that the singularity will remain as long as there is a finite drop in stress at the crack tip[22].

‡It is this zone in our model that corresponds to the Barenblatt cohesive fracture zone, not the entire craze.

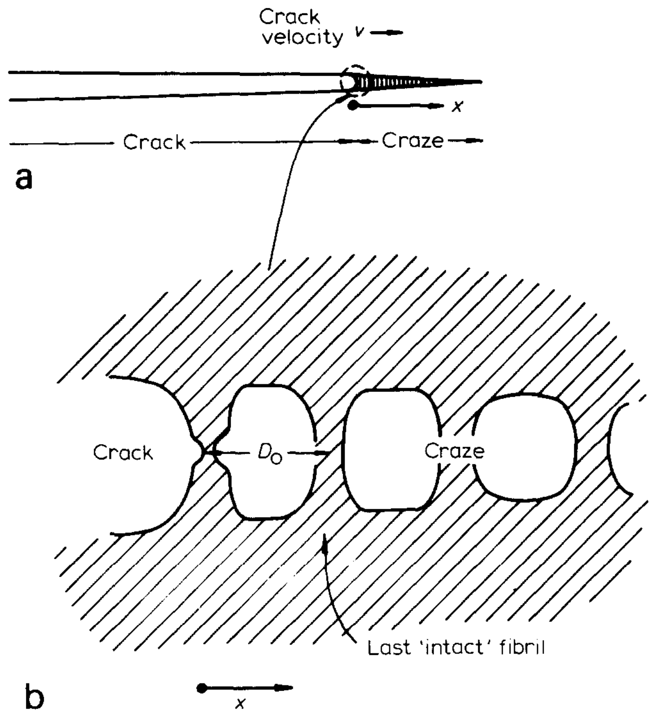


Figure 2 (a) Schematic of craze at the tip of a semi-infinite crack moving at a steady velocity v. (b) Magnified view of the craze at the crack tip. Last loadbearing fibril is one fibril spacing D_0 ahead of the crack tip

the so-called self stresses σ^P of the plastic zone. These stresses are those that would exist if the displacements of the boundary of the zone were fixed and the applied load was removed from the sample.

These self stresses for a craze are most conveniently expressed in terms of a linear dislocation density $\alpha(x)$,

$$\alpha(x) = -2\frac{dw}{dx}$$

It has to be emphasized that the actual structure of the craze is not a dislocation array. Rather the elastic stresses outside and on the boundary of the craze are the same whether the surface displacement profile $w(x)$ is produced by a dislocation array or by an array of some other structural element. Representing the craze as a distribution of dislocation allows the full power of dislocation theory[24] to be used in the solution of this problem.

The self stress due to the accumulated dislocations $\alpha(a)$ at the crack tip has been found previously[22,23] to be

$$\sigma^P(x) = -\frac{E^*}{4\pi}\frac{1}{\sqrt{x}}\int_{CR}\frac{dx_1\alpha(x_1)}{\sqrt{x_1}} + \frac{E^*\sqrt{x}}{4\pi}\int_{CR}\frac{dx_1\alpha(x_1)}{\sqrt{x_1}(x-x_1)} \tag{4}$$

where x_1 is a coordinate along the craze and $\int_{CR}$ is an integral over the whole craze length. The first term on the right hand side of equation (4) has an inverse square root singularity at the crack tip ($x \rightarrow 0$) whereas the second term is non-singular as long as $\alpha(x)$ is continuous over the craze and goes to zero at the craze tip.

If we define a plastic stress intensity factor

$$K_p = -\frac{E^*}{2\sqrt{2\pi}}\int_{CR}\frac{dx_1\alpha(x_1)}{\sqrt{x_1}} \tag{5}$$

then the singular part of σ^P is of the form,

$$\sigma^P_{sing} = K_p/\sqrt{2\pi x} \tag{6}$$

The total tensile stress normal to the craze surfaces is just $\sigma = \sigma^A + \sigma^P$ so that

$$\sigma = K/\sqrt{2\pi x} + \frac{E^*\sqrt{x}}{4\pi}\int_{CR}\frac{dx_1\alpha(x_1)}{\sqrt{x_1}(x-x_1)} \tag{7}$$

where

$$K = K_A + K_p \tag{8}$$

Thus the total stress has a singularity that may be characterized by a *local* stress intensity factor K which is simply the sum of an applied stress intensity factor K_A and a plastic stress intensity factor K_p, due to the displacements produced by the craze. Since K_p is usually negative (for monotonic loading, at least), one can imagine K_p as screening K_A to produce a smaller local K at the crack tip. One can now begin to see the problems inherent in using the Dugdale model which assumes K_p always cancels K_A.

For a moving crack K_p, and thus K, are functions of v, since craze growth like all plastic deformation, is time dependent. On the other hand, in view of our previous discussion on the crack extension force G, which is now given by $G = K^2/E^*$, the (local) value of K should control the crack velocity. To proceed further on this problem we must first solve two subproblems: (1) find the dependence of K_p on crack velocity

$$K_p = K_p(v) \tag{9}$$

and (2) determine a reasonable form for the dependence of crack velocity on the local K,

$$v = v(K) \tag{10}$$

THE PLASTIC STRESS INTENSITY FACTOR OF A MOVING CRACK

We now compute the value of K_p

$$K_p = -\frac{E^*}{2\sqrt{2\pi}}\int_{CR}\frac{dx_1\alpha(x_1)}{\sqrt{x_1}} \tag{11}$$

for a moving crack. In general the integral will depend on past history, all the previous rates of widening at all points along the craze for example. If we seek only steady solutions, those for which the crack v = constant, the history dependence disappears and the problem is greatly simplified. Steady state means that

$$\dot{\alpha} + v\frac{\partial\alpha}{\partial x} = 0 \tag{12}$$

where $\dot{\alpha}$ is the time rate of change of α at a fixed point in the fixed coordinate frame $x_0 = x + vt$. Consider the following integral I

$$I \equiv \frac{1}{v} \int_{CR} dx_1 \sqrt{x_1} \dot{\alpha}(x_1) \quad (13)$$

Substituting $\dot{\alpha}(x_1)$ from equation (12) to give

$$\frac{1}{v} \int_{CR} dx \sqrt{x_1} \dot{\alpha}(x_1) = -\int_{CR} dx_1 \sqrt{x_1} \frac{\partial \alpha}{\partial x_1} \quad (14)$$

and integrating the right hand side of equation (14) by parts, we find

$$\frac{1}{v} \int_{CR} dx_1 \sqrt{x_1} \dot{\alpha}(x_1) = \sqrt{x_1} \alpha(x_1) \Big|_0^{CR\,tip} + \frac{1}{2} \int_{CR} \frac{dx_1 \alpha(x_1)}{\sqrt{x_1}} \quad (15)$$

Since $\alpha(x_1)$ is at most logarithmically singular at $x_1 = 0$ (see equation (1)) and must vanish at the craze tip (CR tip), the first term on the right hand side of equation (15) is identically zero and by substitution into equation (5),

$$K_p = -\frac{E^*}{\sqrt{2\pi}} \frac{1}{v} \int_{CR} dx_1 \sqrt{x_1} \dot{\alpha}(x_1) \quad (16)$$

K_p now depends only on current deformation rates ($\dot{\alpha}$'s) and on the steady crack velocity.

To go further we must propose a kinetic law for craze widening. Current experimental evidence strongly indicates that crazes widen and the fibrils lengthen, by a surface drawing process, analogous to widening of a neck in a textile fibre. For a fibre drawing at constant extension ratio λ, the displacement rate of its shoulders is just

$$\dot{w} = (\lambda - 1)\dot{\tau}/2 \quad (17)$$

where $\dot{\tau}$ is the widening rate of the neck. We believe an analogous equation holds for the displacement rate of the craze boundaries. Logically the displacement rate must increase with the tensile stress acting on the craze boundary in much the same way as the drawing rate of a textile fibre increases with tensile stress[25–27]. An appropriate equation for the stress dependence of $\dot{w}$ is a power law

$$\dot{w} = \dot{w}_y (\sigma/\sigma_y)^n \quad (18)$$

where $\dot{w}_y$ is displacement rate at a reference flow (drawing) stress σ_y.† We may now compute $\dot{\alpha}(x)$ from equation (18), and $\dot{\alpha} = -2\,d\dot{w}/dx$, to be

$$\dot{\alpha} = -\frac{2n\,\dot{w}_y}{\sigma_y} (\sigma/\sigma_y)^{n-1} d\sigma/dx \quad (19)$$

From equation (16) we calculate the plastic stress intensity factor.

$$K_p = -\frac{2E^* n\,\dot{w}_y}{\sqrt{2\pi}\sigma_y v} \int_{CR} dx_1 \sqrt{x_1} (\sigma/\sigma_y)^{n-1} (d\sigma/dx_1) \quad (20)$$

Given the strong dependence of $\dot{w}$ on σ (n is usually quite large), it seems reasonable to approximate σ by the singular term in equation (7), i.e.,

$$\sigma(x_1) = K/\sqrt{2\pi x_1} \quad (21)$$

This approximation, which is the major one in the theory, is justified in detail in the discussion.

Equation (21) can now be transformed from an integral over distance ahead of the crack tip to an integral over the normalized stress $\eta = \sigma/\sigma_y$ to yield

$$K_p = -\frac{E^* n \dot{w}_y K}{v\pi\sigma_y} \int_0^{\eta_L} d\eta\, \eta^{n-2} \quad (22)$$

where η_L is the upper limit to normalized stress to be discussed in the next section.* The result of integration is

$$K_p = -\frac{K}{v} \frac{E^*}{\pi} \frac{\dot{w}_y}{\sigma_y} \frac{n}{n-1} (\eta_L)^{n-1} \quad (23)$$

THE LIMITING STRESS

Inspection of equation (22) reveals the need for a limiting stress. If the upper limit of the integral is extended to $\eta = \sigma/\sigma_y = \infty$, the integral diverges for all physically reasonable values of n. Fortunately there are good physical reasons why η_L cannot be infinite. An upper bound for η_L can be derived from the stress at which $\dot{w}$ from equation (18) equals c_s the speed of sound in the material. At stresses above this value $\dot{w}$ must be constant at about c_s and $\dot{\alpha} = 0$. While this bound is perfectly reasonable mathematically, it is much larger than values of η_L that can be inferred from experiment. Some other mechanisms must limit η_L to much smaller values than the η_L derived from the sound speed criterion.

One possible mechanism is suggested by examining *Figure 2b*, which shows a highly magnified schematic drawing of the craze at the crack tip. The fibril immediately at the crack tip is undergoing the rapid terminal stages of local creep failure as discussed in more detail in the next section. The next fibril to the right, at $x = D_0$, the fibril spacing, is the last craze fibril to be able to bear the full stress implied by the local value of the stress intensity factor K, i.e.,

$$\sigma(D_0) = K/\sqrt{2\pi D_0} \quad (24)$$

Since there are no intact fibrils closer to the crack tip than D_0, the displacement rate $\dot{w}$ due to fibril drawing in this region $[0 < x < D_0]$ must be effectively zero. Hence

†One might argue that a stress activated rate equation of the form $\dot{w} \propto \exp(V^*\sigma/2k_B T)$ would be more appropriate. Since experimental values of $V^*/2k_B T$ are usually large, it is difficult to distinguish this form from a power law with large n; the latter has the advantage of mathematical simplicity. One must also expect the power law to break down at low and high σ. At low σ, below a finite stress σ_0, $\dot{w} \to 0$ since the free energy of the deformed glass is higher than that of the undeformed glass[27]. At high σ, $\dot{w}$ ultimately must not exceed the speed of sound in the material. Under the practical fibril drawing conditions involved in crazing ahead of a constant velocity crack neither of these conditions is important.

* Equation (22) would appear to diverge at the lower limit $\eta = 0$ for $n < 1$. Since $\dot{w}$ should vanish at a finite $\sigma = \sigma_0$[27] and since n is usually much greater than one anyway, this apparent divergence does not raise practical difficulties.

$$\eta_L = \sigma(D_0)/\sigma_y \quad (25)$$

or

$$\eta_L = K/K_y \quad (26)$$

where $K_y = \sigma_y\sqrt{2\pi D_0}$.
Using this limit K_p from equation (23) becomes

$$K_p = -E^*\sqrt{\frac{2D_0}{\pi}}\frac{\dot{w}_y}{v}\frac{n}{n-1}\left(\frac{K}{K_y}\right)^n \quad (27)$$

THE CRACK VELOCITY

We must now find a relationship between the local K and the crack velocity. Several possibilities exist. In his original paper on creep crack growth Hart[23] proposed a semi-brittle crack growth law of the form

$$v = v^* \sinh\left(\frac{G - G_0}{G_1}\right) \quad (28)$$

where G_0 is a minimum crack extension force for crack growth (from Griffith[28] $G_0 = 2\gamma$, twice the surface energy), $G_1 = k_B T/A^*$ and $v^* = v_0 \exp(-\Delta G_0^*/k_B T)$ where A^* and ΔG_0^* are an activation area and free energy for crack tip bond rupture, respectively. Here k_B is Boltzmann's constant and v_0 is a crack velocity based on an attempt frequency f of bond rupture (i.e., $v_0 = f/a_0$, where a_0 is the spacing between atoms). While this semi-brittle law is appropriate for certain metals (steel, for example), it may not apply to the rupture of craze fibrils since there is considerable evidence that these fail during crack growth by a ductile, local creep mechanism[1,10,29]. Nevertheless it should be understood that equation (28) and particularly G_0, sets a lower limit for G below which the crack velocity must approach zero, and K_p must tend to infinity.

A model for the crack growth by local creep failure of fibrils is suggested by the recent paper of Trassaert and Schirrer[29]. They define a failure time t_f for the craze fibrils as being the craze length divided by the crack velocity and show that this failure time is a function of the average stress over the craze as determined from the Dugdale model. We adopt a similar approach here, only we take into account the effect of the variation of local true stress $\hat{\sigma}$ along the craze.†

One would expect that the failure time must satisfy the following equation

$$\int_{CR} \frac{dt}{t_f(\hat{\sigma})} = 1 \quad (29)$$

which simply expresses the idea that creep damage is cumulative. Since at steady state $dt = -dx/v$, equation (29) may be rewritten as

† While it may seem contradictory to assume the craze displacements are due to fibril drawing but that creep dominates fibril failure, the creep we envision is highly localized to a 'weak' entanglement transfer length[10] along the length of the fibril. While the local creep strain in such a short length (~40 nm) may be large, its overall contribution to the craze displacement prior to fracture is negligible.

$$v = -\int_{CR} \frac{dx}{t_f(\hat{\sigma})} \quad (30)$$

It is common for the creep failure time at constant stress to obey the following law

$$\frac{t_f(\hat{\sigma}_1)}{t_f(\hat{\sigma}_2)} = \frac{\dot{\varepsilon}(\hat{\sigma}_2)}{\dot{\varepsilon}(\hat{\sigma}_1)} \quad (31)$$

i.e., the failure time is inversely proportional to the creep strain rate $\dot{\varepsilon}$. This law expresses the assumption that the damage processes in creep (e.g., chain disentanglement in local regions) scale with the creep strain rate. It seems reasonable, and is mathematically convenient, to represent the dependence of $\dot{\varepsilon}$ on the true stress in the fibrils by a power law, which gives

$$t_f(\hat{\sigma}) = \frac{1}{\dot{\varepsilon}_c}\left(\frac{\hat{\sigma}_c}{\hat{\sigma}}\right)^m \quad (32)$$

where $(\dot{\varepsilon}_c)^{-1}$ is the fibril failure time at a reference stress $\hat{\sigma}_c$.

The finely divided nature of the craze dictates a lower limit for the applicability of equation (32) due to the surface tension γ of the fibrils[30]. A fibril of diameter $\bar{D}$ can only elongate by creep if $\hat{\sigma} > \hat{\sigma}_m$ where

$$\hat{\sigma}_m = 2\gamma/\bar{D} \quad (33)$$

and strictly speaking equation (32) should be modified by subtracting off $\hat{\sigma}_m$ from both $\hat{\sigma}_c$ and $\hat{\sigma}$. However, since crazes only grow in width if

$$\hat{\sigma}\bar{D} = 8\Gamma\lambda^{1/2}/\beta \gg \hat{\sigma}_m\bar{D}$$

where β is a constant of order unity, and where Γ is a surface energy which takes into account chain scission at fibril surfaces $(\Gamma > \gamma)$[10], the existence of a lower limit for equation (32) is not of practical importance and will be ignored. Nevertheless we note that $\hat{\sigma}_m$ provides formal counterpart to Hart's G_0, a local stress and stress intensity factor $K_m = \hat{\sigma}_m\sqrt{2\pi D_0}$ below which fibrils will not fail and cracks will not propagate.

The final step is to transform the integral in equation (30) from an integral over space to an integral over craze stress σ. Since the true fibril stress $\hat{\sigma} = \lambda\sigma$ and $dx = -1/\pi(K^2/\sigma^3)d\sigma$, equation (30) becomes

$$v = \frac{\dot{\varepsilon}_c K^2}{\pi}\int_0^{\sigma_L} \frac{d\sigma}{\sigma^3}\left(\frac{\sigma}{\sigma_c}\right)^m \quad (34)$$

where $\sigma_c = \hat{\sigma}_c/\lambda$. Finally we recognize that a divergence in velocity occurs if we integrate to $\sigma = \infty$ for practical values of m $(m \geq 2)$, a divergence that is similar to the one in K_p noted above. We solve this problem in the same way, by using the same upper limit stress $\sigma(D_0)$ defined in equation (25). With this limit the velocity becomes

$$v = \frac{\dot{\varepsilon}_c K^2}{(m-2)\pi}\frac{[\sigma(D_0)]^{m-2}}{\sigma_c^m} \quad (35)$$

or

$$v = v_c\left(\frac{K}{K_c}\right)^m \quad (36)$$

where $K_c = \frac{\hat{\sigma}_c}{\lambda}\sqrt{2\pi D_0}$ and $v_c = 2\dot{\varepsilon}_c D_0/(m-2)$.

THE LOCAL K *VERSUS* K_A RELATION

We may now write an equation relating the applied stress intensity factor K_A to the local stress intensity factor K using equations (8), (27) and (36). The result is

$$K_A = K + B/K^{(m-n)} \quad (37)$$

where

$$B = \frac{E^* \dot{w}_y}{\dot{\varepsilon}_c} \frac{n(m-2)}{(n-1)} \frac{(\sqrt{2\pi D_0})^{(m-n-1)}}{(\sigma_y)^n} \left(\frac{\hat{\sigma}_c}{\lambda}\right)^m$$

or

$$B = \frac{K_E \dot{w}_y}{v_c} \frac{n}{n-1} \frac{K_c{}^m}{K_y{}^n}$$

where $K_E = E^*\sqrt{2\pi D_0}$.

We note once more that the quantity K_A is the usual experimental parameter, usually designated simply as K, that describes the apparent stress intensity factor under the assumption of linear elasticity. The quantity designated here as K is the local stress intensity factor at the crack tip and is the source of the crack propagation force.

DISCUSSION

For this theory to make physical sense m must be greater than n.* Under these conditions a plot of K *versus* K_A yields a C-shaped curve opening to larger K_A as shown in *Figure 3*. To interpret this curve it is useful to remember that steady state crack growth (constant v) corresponds to $\dot{K} = 0$. To the right of this curve lies the region $\dot{K} > 0$ and to the left of this curve the region $\dot{K} < 0$ as illustrated in *Figure 3a*. We may consider the curve itself to have two branches, meeting at the 'nose', point t.

Consider points s_1, s_2 on a line of constant K_A near the upper branch of the curve as shown in *Figure 3b*. Point s_1 is below the upper branch, in the region where $\dot{K} > 0$, or K is increasing with time. As K increases the system tends towards the point s on the upper branch. Point s_2 is above the upper branch, in the region where $\dot{K} < 0$, and K thus decreases with time. In this case the system (at constant K_A) also tends to point s on the upper branch. The upper branch thus represents a curve of stable, steady state crack growth. Points that are displaced from the branch have a tendency to return to the branch. Topologically, this branch of the curve is a valley.

Now consider points u_1, u_2 on the same line of constant K_A near the lower branch of the curve. Point u_1 is below

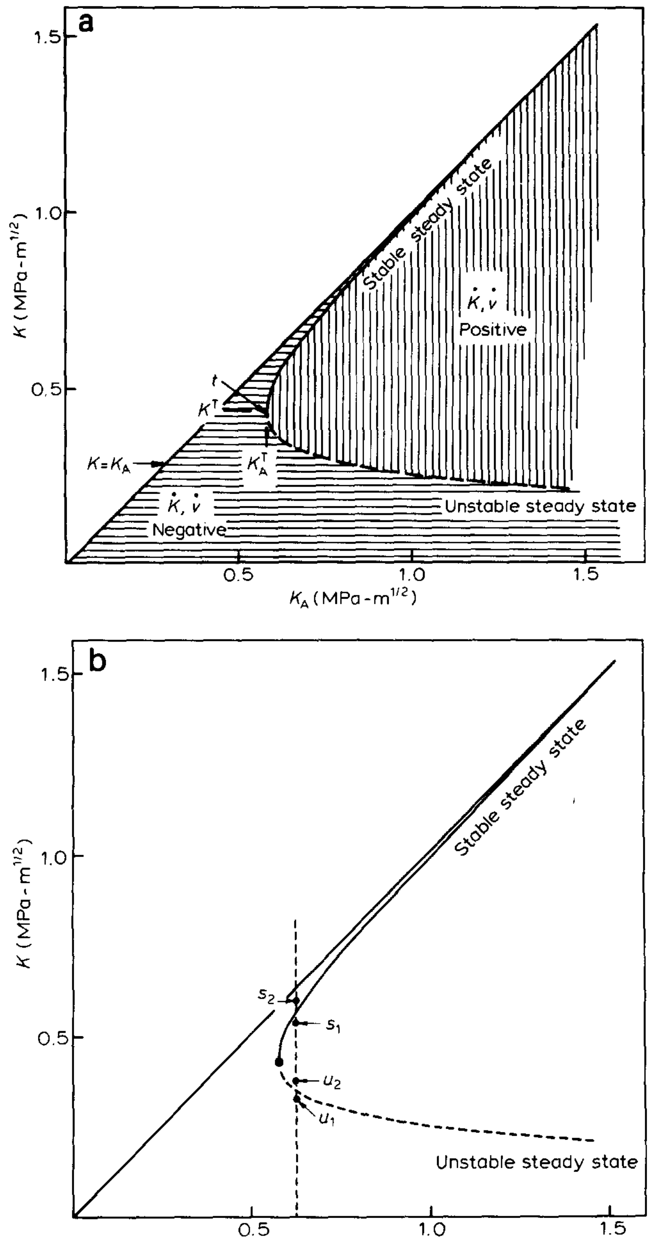


Figure 3 (a) Local stress intensity factor K *versus* applied stress intensity factor K_A for a crack moving at a steady velocity. The values shown correspond to B=0.012 (MPa-m$^{1/2}$)^4 and m-n=3

the lower branch in the region $\dot{K} < 0$ where K decreases with time. Thus as time progresses the crack starting from this point will move away from the lower branch, K will approach the minimum value K_m for crack growth and the crack will stop. At point u_2, it is quite another story. This point is in the region $\dot{K} > 0$ and K increases with time. The crack at point u_2 thus increases its K and v, and moves up away from the lower branch at constant K_A. Although the points on the lower branch of the K vs. K_A curve are steady state values, they represent an unstable steady state. Topologically this branch represents a ridge.

It is now clear that the point t at the nose where both branches meet is a very special point. It represents the minimum conditions of K_A and K (or v) for stable, steady state crack growth. There are two very important consequences that follow from the existence of this point:

* If the stress dependence of fibril drawing is greater than that of fibril failure, the craze will simply widen indefinitely without fibril failure or stable steady state crack growth; K becomes a smaller and smaller fraction of K_A as K_A increases.

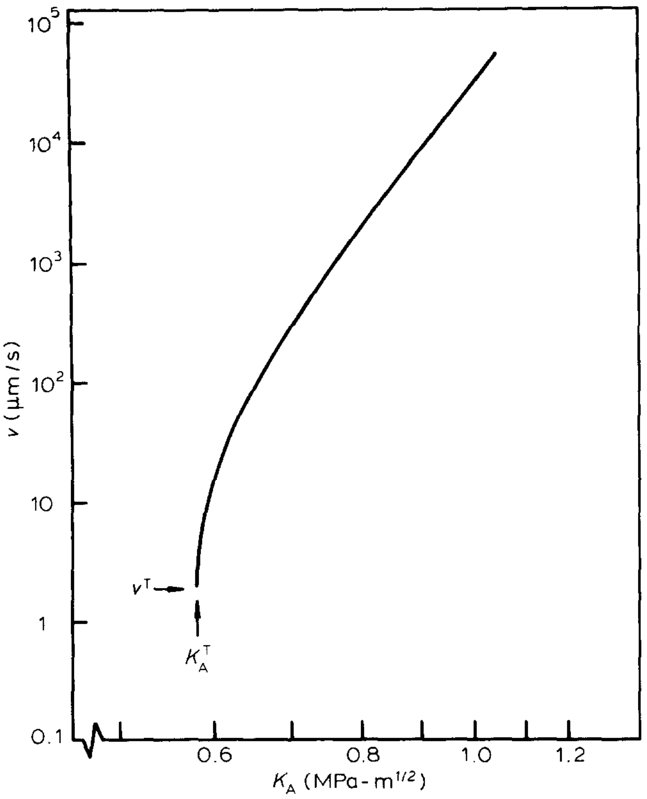


Figure 4 Crack velocity *v versus* applied stress intensity factor K_A for the stable branch of the curve in *Figure 3* with $m=12$. Note the vertical tangent at the threshold

(1) There must exist a threshold applied $K_A = K_A{}^T$, a threshold local $K = K^T$, and a threshold crack velocity $v^T = v_c(K^T/K_c)^m$, below which (stable) steady state crack propagation is not possible. The most significant of these predictions is that of a non-zero threshold velocity. If we try to propagate a crack at $v < v^T$, it will not grow in a stable manner.

(2) At the nose of the curve, the slope dK/dK_A is infinite. This means at the threshold that dv/dK_A is infinite. However, at very large K_A, the second term on the right hand side of equation (37) is negligible compared to the first, $K \rightarrow K_A$ and $v \rightarrow v_c(K_A/K_c)^m$. A plot of log v vs. log K_A must have an infinite slope at the threshold which then decreases smoothly to a slope approaching m for $K_A \gg K_A{}^T$, as shown in *Figure 4*.

Before proceeding further it is reasonable to check whether these predictions are obeyed even qualitatively by polymer glasses. Unfortunately most modern fracture testing utilizes displacement controlled tests. (For example a screw driven machine is often used to load a precracked specimen at a constant displacement rate and the crack growth and load is monitored.) Even with specimens where K_A is constant with crack length (double torsion samples for example) it is difficult to know whether any crack growth (particularly at low v) is truly steady state or not. It is important to emphasize that the theory only gives results for steady state crack growth. Cracks loaded to K_A below the threshold $K_A{}^T$ still grow initially but *eventually* they will stop. The only experiments which are foolproof in this regard are those on constant K_A vs. crack length specimens where constant load is applied and where the crack velocity is monitored over a large increment of crack growth to be sure that it is constant.

Such experiments have been carried out by Aleshin, Aero, Lebedeva and Kuvshinskii[31] on various samples of poly(methylmethacrylate) (PMMA) and (crosslinked) copolymers of methylmethacrylate with ethylene glycol dimethacrylate. Some of their results are shown in *Figure 5*. Note the existence of a threshold $K_A{}^T$ and a threshold velocity v^T below which the crack will not propagate. Note also that the v *versus* K_A curves approach the threshold with a nearly vertical slope.*

Using the experimental values of $K_A{}^T$ and v^T, equation (37) may be rewritten to obtain

$$\frac{K_A(v)}{K_A{}^T} = \frac{p}{1+p}\left(\frac{v}{v^T}\right)^{1/m} + \frac{1}{1+p}\left(\frac{v^T}{v}\right)^{p/m} \tag{38}$$

where $p = m - n$. The exponent m may be determined from the slope of the log v *versus* log K_A line at $v \gg v^T$ where only the first term of equation (38) contributes appreciably and $p = m - n$ may be found from the extrapolation $K_A{}^E$ of this line to $v = v_T$ since $K_A^E/K_A^T = p/(1+p)$. The solid lines in *Figure 5* represent the theory (equation (38)) with the parameters determined in this way. That the fit is excellent is perhaps not surprising in view of the number of parameters used for the fit but nevertheless these all have reasonable values, e.g., $m > n > 1$.

The threshold parameters $K_A{}^T$ and v^T may be expressed in terms of the other material parameters as follows:

$$K_A{}^T = \frac{1+p}{p}\left(\frac{np}{n-1}\frac{K_E \dot{w}_y}{v_c}\right)^{1/(p+1)} K_c^{m/(p+1)} K_y^{-(m-p)/(p+1)} \tag{39}$$

$$v_T = v_c\left(\frac{np K_E \dot{w}_y}{(n-1)v_c}\right)^{m/(1+p)} K_c^{-mp/(1+p)} K_y^{-(m-p)/(p+1)} \tag{40}$$

A decrease in K_y produced by decreasing either σ_y or D_0 should lead to increases in both $K_A{}^T$ and v^T. On the other hand, a decrease in K_c produced by decreasing σ_c or D_0 should lead to a decrease in $K_A{}^T$ but an increase in v^T. These changes are what one would expect intuititively. Experimentally[31], decreases in $K_A{}^T$ are produced by decreasing molecular weight and by adding a comonomer crosslinking agent, ethylene glycol dimethacrylate[31]. An increase in $K_A{}^T$ is produced by adding small amounts of plasticizer[31]. It does not seem worthwhile at this time, however, to attempt to extract K_c and K_y for these various samples since the exponents m and p are also observed to change somewhat with molecular weight, crosslinking and plasticizer content.

We may now also assess the accuracy of the initial assumption that the non-singular term of equation (7) is negligible. This point is important since we have actually assumed subsequently a limiting stress at the crack tip,

$$\sigma(D_0) = K/\sqrt{2\pi D_0}$$

To estimate how serious the error introduced is, let us assume the non-singular stress is a constant value $\langle\sigma\rangle$

*In fact the empirical expression proposed by Aleshin *et al.*, namely $K_A^2(v) = (K_A^T)^2 + (K^*)^2[\ln v/v^t]^2$, to represent for their results has just such a vertical tangent at the threshold.

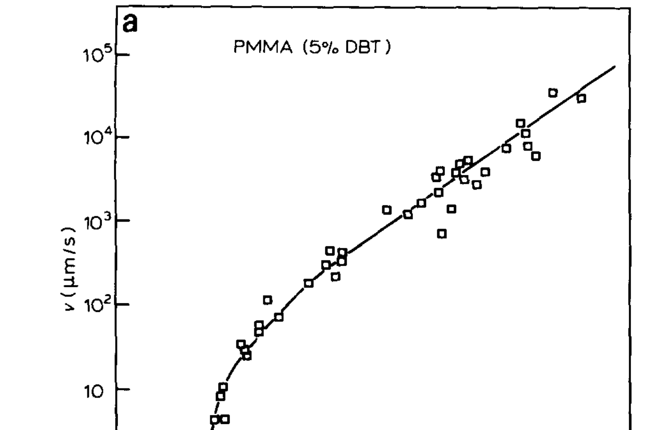


Figure 5 (a) Crack growth data for PMMA plasticized with 5% dibutylphthlate (DBT) from ref. 31. Solid line represents the theory with $m=12$ and $n=9$. (b) Crack growth data for a crosslinked methylmethacrylate - 5% ethylene glycol dimethacrylate copolymer from ref. 31. Solid line represents the theory with $m=19$ and $n=14$

over the craze which is of finite length Δa. If the non-singular term may be ignored the following inequality must hold,

$$\frac{\langle\sigma\rangle}{\sigma(D_0)} \ll 1$$

We can compute $\langle\sigma\rangle$ from the Dugdale model since it would be the plastic zone stress in this model at an applied $K_A{}^D$ equal to the actual K_p. Under these conditions

$$\langle\sigma\rangle = \frac{\pi}{2}\frac{K_A{}^D}{\sqrt{2\pi\Delta a}} = \frac{\pi}{2}\frac{K_p}{\sqrt{2\pi\Delta a}}$$

Thus the ratio $\langle\sigma\rangle/\sigma(D_0)$ becomes

$$\frac{\langle\sigma\rangle}{\sigma(D_0)} = \frac{\pi}{2}\frac{K_p}{K}\sqrt{\frac{D_0}{\Delta a}}$$

On the stable steady state curve K_p has its maximum value

$$K_p^T = K^T/(m-n)$$

at the threshold. In addition since $D_0 \simeq 0.1$ μm and $\Delta a \simeq 100$ μm,

$$\frac{\langle\sigma\rangle}{\sigma(D_o)} < \pi/200(m-n)$$

For all reasonable values of m and n the right hand side of this inequality is much less than one. The assumption that the non-singular part of the stress may be ignored is thus quite good.

The non-steady state behaviour to be expected from this model, while it cannot be computed exactly, has interesting qualitative features that are apparent from *Figure 3*. The plastic stress intensity factor K_p is represented on this diagram as the difference between actual K, K_A point describing the state of the crack at a time t and the line $K=K_A$. Imagine the following step loading thought experiment. The crack at $t=0$ is loaded (infinitely rapidly) to $K_A=K_A^\circ$ which is below $K_A{}^T$ and then held at K_A° for a certain (short) time t_s. As illustrated in *Figure 6a*, during this time a craze plastic zone grows, K_p increases, K decreases and the crack slows since we expect the relation $v=v_c(K/K_c)^m$ to still hold approximately under these non-steady state conditions. After t_s has elapsed, the crack is loaded, again infinitely rapidly, to K_A on the stable branch of the K vs. K_A curve. The crack now should grow at its steady state velocity $v(K_A^P)$.

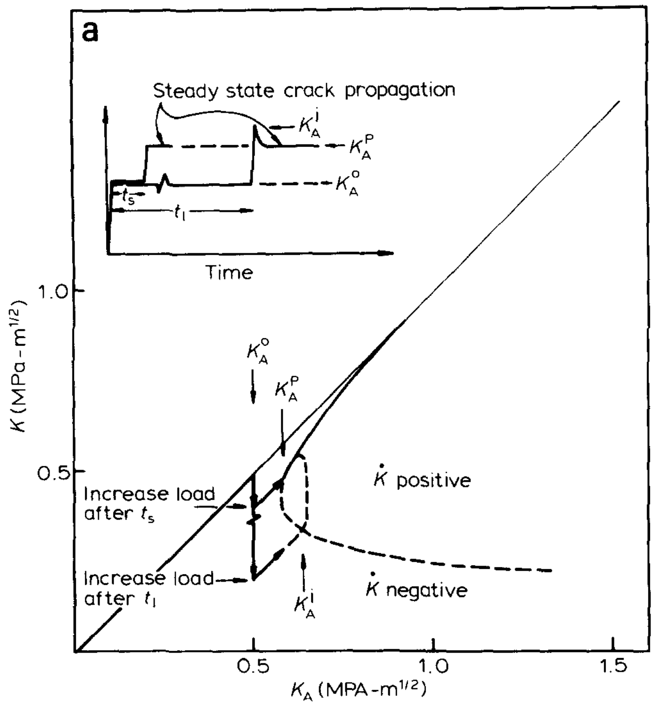
Now suppose we hold the crack for a much longer time t_l at K_A°. K_p continues to increase, and K and v continue to decrease. After time t_l the crack is loaded, infinitely rapidly again along a line K_p=constant, to $K_A{}^P$ as shown in *Figure 6a*. But now the state of the crack (K, K_A) is below the unstable branch of the K vs. K_A steady state crack growth curve. If held at $K_A{}^P$, K and v will continue to decrease, and K_p to increase until the crack stops. To achieve steady state crack propagation in this latter case K_A must transiently increase to at least beyond $K_A{}^i$, the point where the loading K_p=constant line intersects the unstable branch of the steady state growth curve. Once inside the region $K>0$, K increases and K_p decreases and ultimately K_A can be reduced to $K_A{}^P$ with ensuing steady state crack growth. Such step loading experiments give one the possibility of determining the existence of the unsteady state branch and its approximate shape.

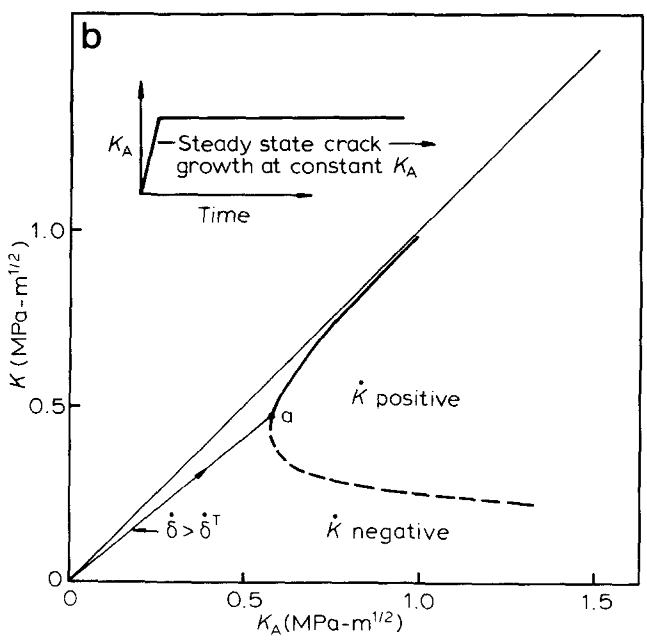
The normal fracture experiments, however, are displacement controlled, not load controlled. Even though so-called constant K_A specimens are used (e.g. the double torsion specimen) for which

$$K_A = \mathscr{L}F \tag{41}$$

where F is the applied force and $\mathscr{L}$ is a compliance calibration constant which is independent of crack length, the specimens are normally loaded with a constant displacement rate $\dot{\delta}$. Under these conditions the rate of change of K_A is given by

$$\dot{K}_A = \frac{\mathscr{L}}{(S+S_m)}\left\{\dot{\delta} - K_A v\left(\frac{2W\mathscr{L}}{E^*}\right)\right\} \tag{42}$$





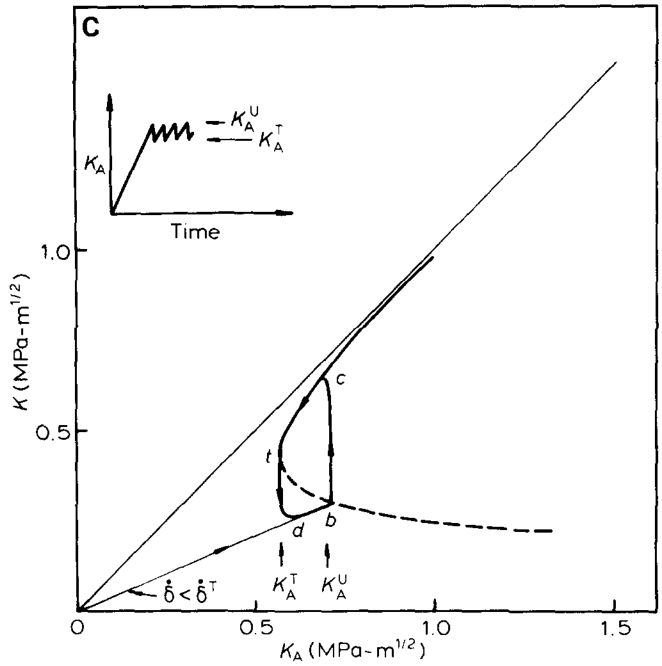


Figure 6 Local stress intensity factor *versus* applied stress intensity factor for: (a) a load history in which the crack is held at $K_A^{\circ} < K_A^{T}$ for various times before rapid further loading. Insert shows K_A histories for short and long holding times, (b) a load history in which the displacement rate of the specimen exceeds the threshold displacement rate. Insert shows the corresponding K_A history, (c) a load history in which the displacement rate of the specimen is much lower than the threshold displacement rate. Insert shows the corresponding K_A history

where S and S_m are compliances of the specimen and the machine respectively and W is the thickness of the specimen along the crack front. For steady state to be attained $\dot{K}_A$ must equal zero which implies that under these conditions,

$$v = \dot{\delta}E^*/(2K_A W \mathscr{L}) \tag{43}$$

As shown above, however, there is a threshold crack velocity v^T, below which the crack will not propagate in a stable steady manner. There exists, therefore, a threshold displacement rate $\dot{\delta}^T$ below which stable crack growth is no longer possible even with constant K_A specimens.

What will occur in such a test? The K vs. K_A diagram in *Figure 6b* supplies the answer. Suppose we load the specimen from $K_A = 0$ at a $\dot{\delta} > \dot{\delta}^T$. Under these conditions the system will travel up a line such as line *oa* on the Figure. While for simplicity we have represented the loading path as a straight line, the actual path probably has a strong downward curvature since we expect K_p to increase more rapidly with time at higher K_A's. In any case the terminal point *a* lies on the stable branch of the steady state K vs. K_A line and once that point is reached, the crack propagates in a stable manner resulting in the K_A vs. δ (or t) curve shown in the insert. The displacement rate can be reduced until $\dot{\delta} = \dot{\delta}^T$ and steady crack growth will occur. The change in the steady K_A level with $\dot{\delta}$ will be negligible since the K vs. K_A and v vs. K_A curves have vertical tangents at the threshold.

Now suppose we load at $\dot{\delta}$ below $\dot{\delta}^T$. The loading path is now that shown in *Figure 6c*. The crack grows slowly in the regime $\dot{K} < 0$ until the unstable branch of the K vs. K_A curve is crossed at *b*. Since the crack is now in the regime $\dot{K} > 0$, it accelerates rapidly and joins the stable branch of the K vs. K_A curve at approximately *c*. However, in this state $2vK_A\mathscr{L}/E^* \gg \dot{\delta}$ so from equation (42) $\dot{K}_A$ is negative. As K_A decreases with time the system follows K vs. K_A curve back toward the threshold point. When the threshold velocity is reached, however, $2v^T K_A W \mathscr{L}/E^*$ is still greater than $\dot{\delta}$, K_A must decrease still more and the system departs from the stable branch, K falling rapidly until the original loading line is regoined at *d*. The cycle now repeats itself giving the unstable K_A vs. δ curve shown in the insert to *Figure 6c*. The maximum value of K_A corresponds roughly to K_A at the point *b* where the unstable branch of the K vs. K_A curve is crossed. The minimum value of K_A corresponds roughly to K_A^T and thus roughly to the K_A at which the crack propagated smoothly at $\dot{\delta}$'s just above $\dot{\delta}^T$. As $\dot{\delta}$ is decreased still further the maximum values of K_A during unstable crack propagation should increase but the minimum value should remain unchanged.

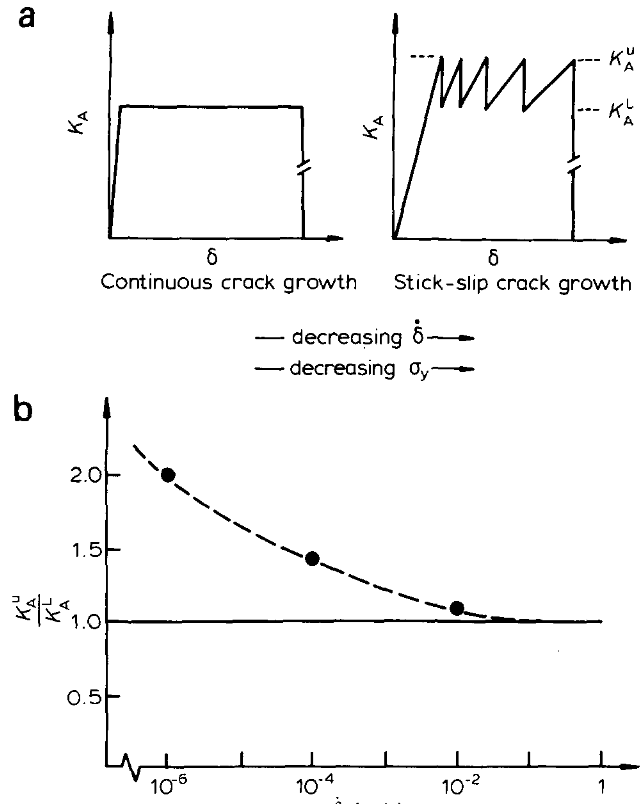


Figure 7 (a) K_A histories typical of continuous crack growth and unstable stick-slip crack growth. Unstable crack growth is promoted by decreasing displacement rate or decreasing flow stress. (b) Ratio of maximum K_A to minimum K_A in unstable crack growth in an epoxy as a function of displacement rate (from ref. 34). The minimum $K_A = K_A{}^L$ is almost independent of $\dot{\delta}$ and is about the same as that for continuous crack growth at higher $\dot{\delta}$'s

Returning to the step loading experiment, now carried out a rapid $\dot{\delta}$ after a long ageing time we should expect to see a transient maximum in K_A vs. δ curve as shown in the insert of *Figure 7a*. This maximum value of K_A should increase with holding time but eventually saturate since the crack cannot relax below the point $K = 0$, $K_A = K_A{}^\circ$.

All the phenomena predicted above are observed in crack growth experiments on epoxies[32–34]*. As shown schematically in *Figure 7a* a transition in crack growth behaviour occurs in these materials as $\dot{\delta}$ is decreased. At high $\dot{\delta}$'s K_A increases steadily until the crack begins to grow at constant $K_A = K_A^L$. As $\dot{\delta}$ is decreased a transition to a so-called slip–stick mode of crack growth occurs. K_A increases to a value $K_A{}^U$ with little of no crack growth, whereupon the crack begins to grow rapidly and K_A drops until the crack arrests when $K_A \simeq K_A{}^L$. K_A now increases again with little or no crack growth until $K_A = K_A{}^U$ and the cycle of 'slip–stick' growth repeats. A plot of $K_A{}^L$ and K_A^U vs. $\dot{\delta}$ in *Figure 7b* shows the transition. This behaviour is clearly that expected from the theory. The critical value of $\dot{\delta} = \dot{\delta}_c$ is just that corresponding to the threshold v^T, i.e. $\dot{\delta}_c = \dot{\delta}^T$. For $\dot{\delta} < \dot{\delta}^T$ stable crack growth is not possible. The lower or crack arrest, value of K_A, K_A^L is approximately equal to the threshold value, K_A^T. The upper, or crack instability value, K_A^U, is approximately the K_A value at which the loading line crosses the unstable branch of the steady state K vs. K_A curve. Furthermore it is observed experimentally[32–34] that any increase in the yield stress of the epoxy, which might be identified with σ_y in the crazing model, produces a rapid increase in $\dot{\delta}^T$. Since v^T depends inversely on σ_y (equation (45)), the increase in $\dot{\delta}^T$ is predicted by the theory.

Likewise in step loading experiments the behaviour of epoxy is in accord with the theory. As shown schematically in *Figure 6a*, reloading the cracked specimen after holding short times at $K_A{}^\circ$, still results in K_A increasing smoothly until the crack propagates at $K_A{}^P$. Holding for longer times at $K_A{}^\circ$ results in a K_A vs. time curve on reloading which increases transiently to $K_A{}^i$, before the crack propagates and K_A decreases to $K_A{}^P$. The holding time necessary for $K_A{}^i$ to exceed $K_A{}^P$ corresponds to the time necessary for $-K_p$ to increase to $-K_p^T$ at the threshold($-K_p{}^T = K_A{}^T - K^T$) assuming rapid loading along a line K_p = constant. Longer ageing times result in the loading line crossing the unstable branch of the steady state K vs. K_A curve at $K_A{}^i > K_A{}^P \simeq K_A{}^T$. However, since $-K_p(t) < K_A{}^\circ$, $K_A{}^i$ should reach a limit at long holding times. (On rapid loading under these circumstances $K = K_A - K_A{}^\circ$.) Such behaviour is seen in *Figure 8*.

Finally it should be noted that the present theory as did that of Hart[23] predicts fatigue crack growth at K_A's that do not exceed $K_A{}^T$. Consider the fatigue cycle shown in *Figure 9* in which K_A is increased to $K_A{}^\circ$ so rapidly that no crack growth occurs, held for time t_0, then loaded rapidly in reverse to $-K_A{}^\circ$, held again for t_0, and so on as a square wave. During the rapid loading $K_p = 0$ and $K = K_A$. For t just greater than zero, K and v are finite, then $-K_p$ increases and K and v fall; if t_0 is long enough, v will eventually fall to zero. Now K_A is reversed to $-K_A{}^\circ$ along the line ab. Again during the rapid reversal $\dot{K}_p = 0$ so $K = -(K_p(t = t_0) + K_A{}^\circ)$. During the holding period bc, the compressive local K is relieved by a reverse plastic deformation of the plastic zone or craze (K_p decays almost to zero in mode I)*. In mode I crack growth does not occur during this portion of the cycle. Upon rapid reloading to $+K_A{}^\circ$ at d, K will increase to just slightly below $K_A{}^\circ$ and v again is finite. In the holding period da, $-K_p$ again increases and K and v fall and the cycle repeats. During the tension ($+K_A{}^\circ$) portion of each cycle a small amount of crack growth occurs before v decreases to zero. The

* While reported observations of crazing in epoxies are equivocal at best, replacing the crazing zone with a zone of shear plasticity and the cracking by local fibril creep with the semibrittle crack growth law of Hart[23] does not change the qualitative argument given here.

* Crack closure prevents K_p from becoming positive in Mode I, but there is no such limitation in either mode II or mode III[23].

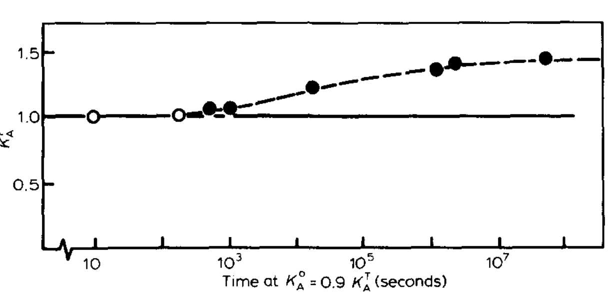


Figure 8 Ratio of maximum stress intensity factor ($K_A{}^i$) to the stress intensity factor for steady state crack growth ($K_A{}^\circ$, 90% of the threshold stress intensity factor for an epoxy (from ref. 34). The K_A histories for short and long holding times are as shown in *Figure 6a*

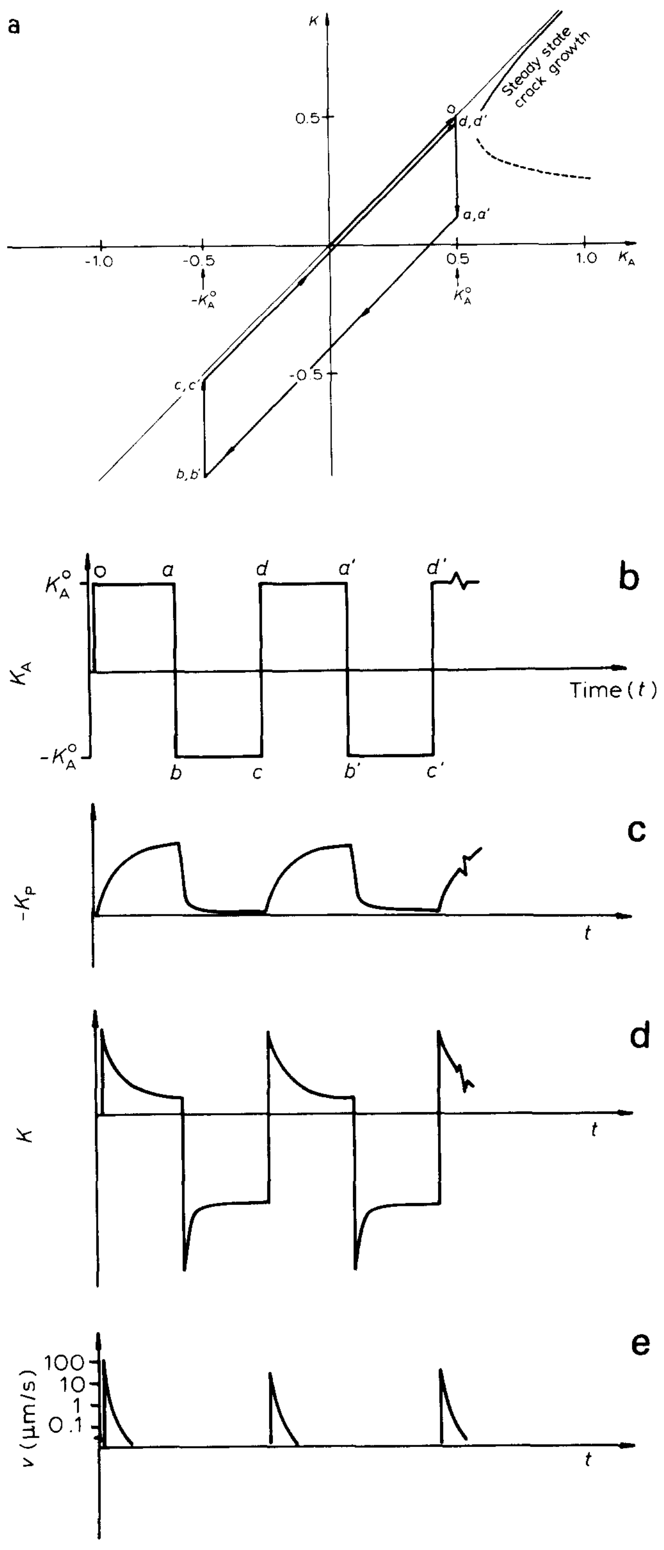


Figure 9 (a) Local stress intensity factor K *versus* applied stress intensity factor for fatigue loading below K_A^T. (b) Square wave K_A fatigue cycle, (c) corresponding K_p cycle, (d) corresponding K cycle, (e) corresponding crack growth velocity cycle in fatigue

reversal of deformations† in the craze (or zone) each cycle results in a transient increase in K on reloading and renewed non-steady state crack growth.

† In the craze the predominant mechanism of plastic deformation during the second cycle is different from that in the first. In the first the craze displacements are due to surface drawing; in the second restraightening of buckled fibrils provides most of the displacement.

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GLOSSARY OF SYMBOLS

A^* activation area for crack growth
a length of crack plus craze
a_0 crack length
a_0 atom spacing
B $= \frac{K_E \dot{w}_y}{v_c} \frac{n}{n-1} \frac{K_c^m}{K_y^n}$
c_s speed of sound in the material
D_0 craze fibril spacing
$\bar{D}$ craze fibril diameter
E^* effective Young's modulus $= \begin{cases} E/(1-v^2) \text{ for plane strain} \\ E \text{ for plane stress} \end{cases}$
F applied force
f attempt frequency of bond rupture
G strain energy release rate or crack extension force
G_I $= k_B T/A^*$
G_{I_c} critical strain energy release rate
G_0 minimum crack extension force for crack growth
K local stress intensity factor at crack tip
$\dot{K}$ time rate of change of local stress intensity factor
K_A applied stress intensity factor
$\dot{K}_A$ time rate of change of applied stress intensity factor
K_A^D applied stress intensity factor for Dugdale model $= K_p$
K_A^E extrapolation of the log K_A vs. log v plot to $v = v^T$
K_A^i transient overshoot of K_A necessary to start crack growth after holding at K_A^o
K_A^L, K_A^U lower and upper values of applied stress intensity factor during slip–stick crack growth
K_A^0 applied stress intensity factor below K_A^T
K_A^P applied stress intensity factor above K_A^T
K_A^T, K_p^T threshold applied stress intensity factor, and threshold plastic stress intensity factor, for stable steady state crack growth
k_B Boltzmann's constant
K_C $= \frac{\hat{\sigma}}{\lambda}\sqrt{2\pi D_0}$
K_E $= E^* \sqrt{2\pi D_0}$
K_m $= \hat{\sigma}_m \sqrt{2\pi D_0}$
K_p plastic stress intensity factor
K_y $\sigma_y \sqrt{2\pi D_0}$
$\mathscr{L}$ compliance calibration constant
m power law exponent for craze fibril creep
n power law exponent for craze displacement rate
p $= m - n$
S_c Dugdale zone tensile stress
S, S_M compliance of specimen and testing machine respectively
$\dot{t}$ widening rate of craze

T absolute temperature
t_f craze-fibril failure time
t_S, t_l holding times at K_A°
V^* shear activation volume
v crack velocity
v^T threshold crack velocity for stable steady state crack growth
v_c $=2\dot{\varepsilon}_c D_0/(m-2)$
v_0 $=f/a_0$
W specimen width along crack front
w craze surface displacement
$\dot{w}$ craze surface displacement rate
w_c critical crack opening displacement
$\dot{w}_y$ craze displacement rate when $\sigma=\sigma_y$
x coordinate measuring distance ahead of crack tip
x_0 distance in front of crack tip in fixed coordinate frame
$\dot{\alpha}$ time rate of change of dislocation density
$\alpha(x)$ linear dislocation density
β constant of order unity
β_c, β $\cos^{-1}(a_0/a)$ and $\cos^{-1}(x/a)$ respectively
Δa craze length
ΔG_0^* activation free energy for crack growth
$\dot{\delta}$ specimen displacement rate
$\dot{\delta}^T$ threshold specimen displacement rate for stable steady state crack growth
$\dot{\varepsilon}$ local craze fibril creep strain rate
$\dot{\varepsilon}_c^{-1}$ craze fibril failute time at $\hat{\sigma}_c$
η normalized stress, σ/σ_y
η_L upper limit to normalized stress $=\sigma(D_0)/\sigma_y$
Γ surface energy with chain scission required
γ surface energy
λ extension ratio of craze fibrils
ν Poisson's ratio
σ^A stress ahead of the crack tip in absence of inelastic deformation
σ_c $\hat{\sigma}_c/\lambda$
σ_L $=\sigma(D_0)$, stress at last intact fibril
σ_0 stress below which $\dot{w}=0$
σ^P self stresses of plastic zone (craze)
σ^P_{sing} singular part of σ^P
σ_y reference drawing (flow) stress
$\hat{\sigma}$ local true stress in craze
$\hat{\sigma}_c$ local craze reference true stress
$\hat{\sigma}_m$ minimum stress for craze fibril creep
$\langle\sigma\rangle$ assumed constant value of stress over craze

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